ALTERNATIVE TO EVOLVING SURFACE FINITE ELEMENT METHOD

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Abstract. ESFEM is a method introduced in [5] in order to solve a linear advection-diffusion equation on an evolving two-dimensional surface with finite elements by using a moving grid with nodes sitting on and evolving with the surface. The evolution of the surface is assumed to be given as a smooth one-parameter family of embeddings of a fixed initial surface into $\mathbb{R}^3$ satisfying uniform C^4 bounds. We calculate an equivalent transformed equation which is defined on the fixed initial surface and can hence be solved numerically on a fixed grid. We present numerical examples which indicate that both approaches are essentially of the same accuracy.

Key words. finite elements, evolving surface, linear advection-diffusion equation

AMS subject classifications. 35K05, 53A05, 65M60

1. Introduction.

In many applications it is important to consider PDEs which are defined on surfaces and not in Euclidean space, especially in the case of parabolic equations it is of interest to assume that these surfaces (where the equation is defined) evolve with respect to time in a certain prescribed way. In [5] the so-called evolving surface finite element method (ESFEM) is proposed in order to solve an advection-diffusion equation on an evolving surface, cf. [5, Sections 1.1 and 1.2]. This setting models e.g. the transport of an insoluble surfactant on the interface between two flowing fluids or pattern formation on the surfaces of growing organisms modeled by reaction-diffusion equations, cf. [5, Section 1.4] for further and a more detailed exposition of applications. There are several papers which deal with linear parabolic equations on evolving surfaces, e.g. in [11, 6] it is shown that classical L^2- and L^∞-estimates carry over to ESFEM and in [12, 8] it is shown that this also holds for error estimates of Runge-Kutta schemes and Backward difference schemes; we also mention [13, 14].

We present the idea of ESFEM according to [5] and note that this method is introduced therein in order to solve a special linear parabolic equation, namely Equation (1.2), with the finite element method.

Let Γ_0 be a smooth, closed, connected and oriented hypersurface in $\mathbb{R}^3$. Let $\Phi(t,\cdot) : \Gamma_0 \to \mathbb{R}^3$, $t \in [0, T_0]$, $T_0 > 0$, be a family of embeddings, Φ smooth, $\Gamma(t) = \Phi(t, \Gamma_0)$ the moving surfaces. We define the set

$$G_{T_0} = \bigcup_{t\in[0,T_0]} \{t\} \times \Gamma(t) \tag{1.1}$$

and consider there the advection-diffusion equation

$$\dot{u} + u\nabla^{\Gamma(t)} \cdot v - \nabla^{\Gamma(t)} \cdot (D_0 \nabla^{\Gamma(t)} u) = 0 \tag{1.2}$$

where $\dot{u}$ denotes the material derivative, $v(p_t, t) = \frac{\partial \Phi}{\partial t}(\Phi(t,\cdot)^{-1}(p_t), t)$ the velocity of the moving surface in a point $(t, p_t) \in G_{T_0}$ and hence $\nabla^{\Gamma(t)} \cdot v$ its tangential divergence. Furthermore, the diffusion coefficient $D_0 : G_{T_0} \to Mat(n+1, \mathbb{R})$ is so that it vanishes

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on the normal space of the moving surfaces, i.e. $D_0(t, p_t)\nu = 0$ for all $\nu \in N_{p_t}$ where N_{p_t} is the normal space of $\Gamma(t)$ in p_t. Equation (1.2) can be written in variational form as

$$\frac{d}{dt}\int_{\Gamma(t)} u\varphi + \int_{\Gamma(t)} D_0 \nabla^{\Gamma(t)} \cdot \nabla^{\Gamma(t)}\varphi = \int_{\Gamma(t)} u\dot{\varphi} \quad \forall \varphi \in C^\infty(G_{T_0}). \tag{1.3}$$

ESFEM considers at every time t an interpolating polyhedral surface $\Gamma_h(t)$ which approximates the evolving surface $\Gamma(t)$ and which consists of triangles with vertices $X_j(t) = \Phi(t, X_j(0))$, $j = 1, ..., N$, sitting on $\Gamma(t)$ and moving with the surface. Here, $X_j(0)$, $j = 1, ..., N$, are fixed nodes on the initial surface $\Gamma(0)$. The finite element basis functions φ_i, $i = 1, ..., N$, are defined on

$$G^h_{T_0} = \bigcup_{t\in[0,T_0]} \{t\} \times \Gamma_h(t) \tag{1.4}$$

and chosen so that $\varphi_i(t, \cdot)$ is piecewise linear (i.e. linear on each triangle of $\Gamma_h(t)$) with $\varphi_i(t, X_j(t)) = \delta_{ij}$.

The fully discrete scheme from [5] in order to solve (1.2) which uses ESFEM can be found in [5, Equation (7.2)] and is as follows. Let $t_0 = 0 < ... < t_M = T_0$, $M \in \mathbb{N}$, be a partition of the time interval, $\Gamma^m_h = \Gamma_h(t_m)$, $u^m = u(\cdot, t_m)$ for a function u on $G^h_{T_0}$ and $u^0_h \in \Gamma_h(0)$ an initial function (and for simplicity D_0 the identity) then we solve for $m = 0, ..., M-1$ the linear system

$$\begin{aligned}&\frac{1}{\tau}\int_{\Gamma^{m+1}_h} u^{m+1}_h \varphi^{m+1}_j + \int_{\Gamma^{m+1}_h} \nabla^{\Gamma^{m+1}_h} u^{m+1}_u \cdot \nabla^{\Gamma^{m+1}_h} \varphi^{m+1}_j \\ =&\frac{1}{\tau}\int_{\Gamma^m_h} u^m_h \varphi^m_j, \quad j = 1, ..., N.\end{aligned} \tag{1.5}$$

Our different approach to solve (1.2) with the finite element method reformulates Equation (1.2) as an equivalent equation on the surface $\Gamma(0)$, cf. Equation (3.13), and uses a fixed grid (the one consisting of the nodes $X_j(0)$, $j = 1, ..., N$,) for the finite element approximation (again an implicit Euler method) which leads to a finite element solution $\tilde{u}^m_h$ defined on $\Gamma_h(0)$. Obviously, both approaches coincide if $\Phi(t, \cdot)$ is the identity, i.e. if there is no motion.

Our aim is to use both approaches, i.e ESFEM and the method which uses the transformed equation, in some example cases and to provide the corresponding error tables.

The paper is organized as follows. In Section 2 we recall some facts about hypersurfaces in the Euclidean space, in Section 3 we derive the reformulated equation, in Section 4 we present our chosen examples and a more detailed description of the parts from which the coefficients of the transformed equation are put together in the implementation. The error tables of the numerical calculations can be found in Tables 1 to 6.

2. Hypersurfaces in $\mathbb{R}^{n+1}$.

We recall some facts and notations of embedded hypersurfaces in $\mathbb{R}^{n+1}$ from [7]. Let $F : \Omega \to \mathbb{R}^{n+1}$ with $\Omega \subset \mathbb{R}^n$ open be a smooth embedding and $M = F(\Omega)$. For $p \in \Omega$ the coordinate tangent vectors $\partial_i F(p) = \frac{\partial F}{\partial p_i}(p)$, $1 \le i \le n$, provide a basis of the tangent space $T_x M$ at $x = F(p)$.

The metric on M is given by

$$g_{ij} = \partial_i F \cdot \partial_j F \tag{2.1}$$

for $1 \le i, j \le n$, the inverse metric by $(g^{ij}) = (g_{ij})^{-1}$.

The tangential gradient of a function $h : M \to \mathbb{R}$ is defined by

$$\nabla^M h = g^{ij} \partial_j h \partial_i F \tag{2.2}$$

where we sum over repeated indices.

For a smooth tangent vector field $X = X^i \partial_i F = g^{ij} X_j \partial_i F$ on M (note that $X_i = X \cdot \partial_i F$) we define the covariant derivative tensor by

$$\nabla_i^M X^j = \partial_i X^j + \Gamma_{ik}^j X^k \tag{2.3}$$

where the Christoffel symbols Γ_{ij}^k are given by

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}). \tag{2.4}$$

The tangential divergence of X on M is defined by

$$\nabla^M \cdot X = \operatorname{div}_M X = \nabla_i^M X^i \tag{2.5}$$

and the Laplace-Beltrami operator of h on M by

$$\Delta_M h = \operatorname{div}_M \nabla^M h = \nabla^M \cdot (\nabla^M h). \tag{2.6}$$

For a smooth vector field $X : M \to \mathbb{R}^{n+1}$ which is not necessarily tangent on M we can also define the divergence with respect to M by

$$\nabla^M \cdot X = \operatorname{div}_M X = g^{ij} \partial_i X \cdot \partial_j F \tag{2.7}$$

which reduces to the above expression if X is tangent on M. We remark that, of course, the divergence in (2.7) is (and transforms like) a scalar function (when changing local coordinates of M). Furthermore, the tangential gradient transforms like a scalar function when changing coordinates in M.

3. Reformulation of the equation on a fixed surface. In this section we derive Equation (3.13) which is an equivalent reformulation of (1.2) on $\Gamma(0)$. The calculation is straightforward but we still present details.

Let $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ be open and $\Phi : \tilde{\Omega} \to \Omega$ a diffeomorphism. The linear differential operator $L : H^2(\Omega) \to L^2(\Omega)$

$$Lu = a^{ij} D_i D_j u + b^i D_i u + cu \tag{3.1}$$

with coefficients $a^{ij}, b^i, c \in L^\infty(\Omega)$, a^{ij} symmetric, transforms into the operator $\tilde{L} : H^2(\tilde{\Omega}) \to L^2(\tilde{\Omega})$

$$\tilde{L}\tilde{u} = \tilde{a}^{ij} D_i D_j \tilde{u} + \tilde{b}^i D_i \tilde{u} + \tilde{c}\tilde{u}, \tag{3.2}$$

i.e. we have $(Lu) \circ \Phi = \tilde{L}\tilde{u}$ for the quantity $\tilde{u} = u \circ \Phi$. Here, by writing $x(\tilde{x}) = \Phi(\tilde{x})$ and $\tilde{x}(x) = \Phi^{-1}(x)$ we set

$$\begin{aligned} \tilde{a}^{ij} &= a^{rs} \frac{\partial \tilde{x}^i}{\partial x^r} \frac{\partial \tilde{x}^j}{\partial x^s} \\ \tilde{b}^k &= b^m \frac{\partial \tilde{x}^k}{\partial x^m} - a^{ij} \frac{\partial^2 x^m}{\partial \tilde{x}^r \partial \tilde{x}^s} \frac{\partial \tilde{x}^r}{\partial x^i} \frac{\partial \tilde{x}^s}{\partial x^j} \frac{\partial \tilde{x}^k}{\partial x^m} \\ \tilde{c} &= c \end{aligned} \tag{3.3}$$

and $\circ\Phi$ operations are suppressed.

These formulas carry over to the surface case when using local coordinates. Let now $\Omega, \tilde\Omega$ be (open subsets of) hypersurfaces in $\mathbb{R}^3$, $\Phi : \tilde\Omega \to \Omega$ a diffeomorphism and let a^{ij}, b^i and c be L^∞-sections of the tensor bundles $T^{2,0}(\Omega)$, $T^{1,0}(\Omega)$ and $T^{0,0}(\Omega)$, respectively, and assume that a^{ij} is symmetric. Let L be defined by

$$Lu = a^{ij}\nabla_i^\Omega \nabla_j^\Omega u + b^i \nabla_i^\Omega u + cu, \quad u \in H^2(\Omega), \tag{3.4}$$

where ∇^Ω denotes the Levi-Cevita connection with Christoffel symbols Γ_{ij}^k on Ω (and $\nabla^{\tilde\Omega}$ and $\tilde\Gamma_{ij}^k$ correspondingly on $\tilde\Omega$) then we have

$$\begin{aligned} Lu =& a^{ij} D_i D_j u + (b^k - a^{ij}\Gamma_{ij}^k) D_k u + cu \\ =& a^{ij} D_i D_j u + \bar b^k D_k u + cu \end{aligned} \tag{3.5}$$

where D_i denote ordinary partial derivatives. Using the formulas (3.3) we get a transformed operator

$$\tilde L \tilde u = \tilde a^{ij} \nabla_i^{\tilde\Omega} \nabla_j^{\tilde\Omega} \tilde u + \tilde b^i \nabla_i^{\tilde\Omega} \tilde u + \tilde c \tilde u, \quad \tilde u \in H^2(\tilde\Omega), \tag{3.6}$$

where now in local coordinates (x^i) of Ω and $(\tilde x^i)$ of $\tilde\Omega$ we have

$$\begin{aligned} \tilde a^{ij} =& a^{rs} \frac{\partial \tilde x^i}{\partial x^r} \frac{\partial \tilde x^j}{\partial x^s} \\ \tilde b^k =& \bar b^m \frac{\partial \tilde x^k}{\partial x^m} - a^{ij} \frac{\partial^2 x^m}{\partial \tilde x^r \partial \tilde x^s} \frac{\partial \tilde x^r}{\partial x^i} \frac{\partial \tilde x^s}{\partial x^j} \frac{\partial \tilde x^k}{\partial x^m} \\ &+ a^{rs} \frac{\partial \tilde x^i}{\partial x^r} \frac{\partial \tilde x^j}{\partial x^s} \tilde\Gamma_{ij}^k \\ \tilde c =& c. \end{aligned} \tag{3.7}$$

A choice of local coordinates in $\tilde\Omega$ induces via Φ local coordinates in Ω and we stipulate that in the following the local coordinates of Ω and $\tilde\Omega$ are related in this way. Then the formulas (3.7) simplify to

$$\begin{aligned} \tilde a^{ij} =& a^{ij} \\ \tilde b^k =& \bar b^k + a^{ij} \tilde\Gamma_{ij}^k \\ \tilde c =& c. \end{aligned} \tag{3.8}$$

Let us consider the case where the main part is in divergence form

$$\begin{aligned} Lu =& \nabla_i^\Omega (a^{ij} \nabla_j^\Omega u) + b^i \nabla_i^\Omega u + cu \\ =& a^{ij} \nabla_i^\Omega \nabla_j^\Omega u + (\nabla_j^\Omega a^{ij} + b^i) \nabla_i^\Omega u + cu \end{aligned} \tag{3.9}$$

then we get the transformed operator with main part in divergence form

$$\begin{aligned} \hat L \tilde u =& \hat a^{ij} \nabla_i^{\tilde\Omega} \nabla_j^{\tilde\Omega} \tilde u + \hat b^i \nabla_i^{\tilde\Omega} \tilde u + \hat c \tilde u \\ =& \nabla_i^{\tilde\Omega} (\hat a^{ij} \nabla_j^{\tilde\Omega} \tilde u) + (\hat b^i - \nabla_j^{\tilde\Omega} \hat a^{ij}) \nabla_i^{\tilde\Omega} \tilde u + \hat c \tilde u \\ =& \nabla_i^{\tilde\Omega} (\hat a^{ij} \nabla_j^{\tilde\Omega} \tilde u) + \check b^i \nabla_i^{\tilde\Omega} \tilde u + \hat c \tilde u \end{aligned} \tag{3.10}$$

where

$$
\begin{aligned}
\hat{a}^{ij} =& a^{ij} \\
\hat{b}^k =& \bar{b}^k + a^{ij}\tilde{\Gamma}^k_{ij} + \nabla^{\Omega}_i a^{ik} = b^k + a^{ij}(\tilde{\Gamma}^k_{ij} - \Gamma^k_{ij}) + \nabla^{\Omega}_i a^{ik} \\
\hat{c} =& c
\end{aligned}
\tag{3.11}
$$

and hence

$$
\check{b}^k = b^k + a^{ij}(\tilde{\Gamma}^k_{ij} - \Gamma^k_{ij}) + \nabla^{\Omega}_i a^{ik} - \nabla^{\tilde{\Omega}}_i a^{ik}. \tag{3.12}
$$

Let us assume that u and Φ are as in Section 1 and that D_0 is the identity. We define $\tilde{u}(t,x) = u(t,\Phi(t,x))$ and transform (1.2) in the equivalent equation

$$
\begin{aligned}
&\frac{d}{dt}\tilde{u} - \nabla^{\Gamma(0)}_i (g^{ij}(t)\nabla^{\Gamma(0)}_j \tilde{u}) \\
&\quad + (g^{ij}(t)(\Gamma^k_{ij}(t) - \Gamma^k_{ij}(0)) + \nabla^{\Gamma(0)}_j g^{kj}(t))\nabla^{\Gamma(0)}_k \tilde{u} + \tilde{u}\nabla^{\Gamma(t)} \cdot v = 0
\end{aligned}
\tag{3.13}
$$

on $[0,T_0] \times \Gamma(0)$ where the time derivative is now a usual partial derivative, $g_{ij}(t)$ denotes the metric and $\Gamma^k_{ij}(t)$ the Christoffel symbols of $\Gamma(t)$. Here, we used that the covariant derivative of the metric vanishes and the coupling of local coordinates via $\Phi(t,\cdot)$.

Remark 3.1. We note that if one considers instead of (1.2) a general linear parabolic equation

$$
\dot{u} - a^{ij}\nabla^{\Gamma(t)}_i \nabla^{\Gamma(t)}_j u + b^i \nabla^{\Gamma(t)}_i u + cu = f \tag{3.14}
$$

on G_{T_0} where

$$
a^{ij} : G_{T_0} \to T^{2,0}(\Gamma(t)), \quad b^i : G_{T_0} \to T^{1,0}(\Gamma(t)), \quad c, f : G_{T_0} \to \mathbb{R} \tag{3.15}
$$

so that $a^{ij}(t,\cdot)$ and $b^i(t,\cdot)$ are sections of the corresponding bundles then the transformation rules (3.11) and (3.12) – of course – provide a reformulation of this equation on the fixed surface as well.

4. Examples and implementation. We choose $\Gamma(0) = \partial B_1(0) \subset \mathbb{R}^3$, $T_0 = \frac{3}{2}$ and define the motion of the surface by

$$
\Phi(t,x) = A(t)x \tag{4.1}
$$

where we consider the cases

$$
A(t) = \begin{pmatrix} \sqrt{1+5\sin t} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.2}
$$

$$
A(t) = \begin{pmatrix} 1+tx_1^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.3}
$$

and

$$
A(t) = \begin{pmatrix} \cos(\eta\mu(x_3)t) & -\sin(\eta\mu(x_3)t) & 0 \\ \sin(\eta\mu(x_3)t) & \cos(\eta\mu(x_3)t) & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.4}
$$

with $\eta = 5$ and $\mu(x_3) = \frac{1}{3}x_3^3$. We append a right-hand side to the equation (1.3) (and correspondingly to (3.13)) and set an initial value which are so that the exact solution of the transformed equation is in all cases $\tilde{u}(t,x) = e^{-6t}x_1x_2$. We calculated the parts from which the three right-hand sides (corresponding to the three cases) can be put together by hand and put them together within the implementation. For the discretization we use the coupling $\Delta t = \Delta x^2$ between the step sizes in time and space.

We point out that the third example is chosen to observe the phenomenon of having only tangential motion which deteriorates the mesh in ESFEM (in the sense that the ratio of the diameter and the incircle radius of a triangle might become large) and of course also affects the coefficients in the transformed equation. We remark that for ESFEM the (relative) mesh size may change and tangential motion of the nodes may deteriorate the mesh. While the former might be compensated by appending additional nodes the latter can be compensated by introducing additional tangential motion of the mesh (ALE-ESFEM), see [9] for the latter.

To be able to input the transformed coefficients as stated in (3.13) into the Distributed and Unified Numerics Environment (DUNE), see [1, 2, 3], we present some auxiliary facts. Let $x = (x^i)_{1\leq i\leq 3}$ be Euclidean coordinates in $\mathbb{R}^3$ and $\xi = (\xi^j)_{1\leq j\leq 2}$ local coordinates in $\Gamma(0)$. Then $x = x^i(\xi^j)$ can be seen as a local representation of the embedding of $\Gamma(0)$ into $\mathbb{R}^3$ and $\Phi(x^i(\xi^j),t)$ of the embedding of $\Gamma(t)$ into $\mathbb{R}^3$. For the latter we drop the time dependence for simplicity and write $\Phi \circ x = \Phi(x^i(\xi^j))$. We have

$$
\begin{aligned}
g_{ij}(t) =& \frac{\partial}{\partial \xi^i}(\Phi\circ x)\cdot \frac{\partial}{\partial \xi^j}(\Phi\circ x)\\
\partial_k g_{ij}(t) =& \frac{\partial^2}{\partial \xi^i \partial \xi^k}(\Phi\circ x)\cdot \frac{\partial}{\partial \xi^j}(\Phi\circ x) + \frac{\partial^2}{\partial \xi^j \partial \xi^k}(\Phi\circ x)\cdot \frac{\partial}{\partial \xi^i}(\Phi\circ x),
\end{aligned}
\tag{4.5}
$$

furthermore,

$$
\nabla_r^{\Gamma(0)} g^{ij}(t) = -g^{ik}(t)g^{lj}(t)\partial_r g_{kl}(t) + \Gamma(0)^i_{rl}g^{lj} + \Gamma(0)^j_{rl}g^{li}. \tag{4.6}
$$

We calculate the tangential divergence of the evolution speed $v = \dot{A}(t)x$

$$
\nabla^{\Gamma(t)}\cdot v = g^{ij}(t)\Big(\frac{\partial}{\partial \xi^i}\dot{A}(t)x + \dot{A}(t)\frac{\partial}{\partial \xi^i}x\Big)\cdot \frac{\partial}{\partial \xi^j}(\Phi\circ x). \tag{4.7}
$$

Let $x = (x^i) \subset \Gamma(0)$ be given. Let $\alpha : \{1,2,3\} \to \{1,2,3\}$ be a bijection so that $|x^{\alpha(1)}| = \min_j |x^{\alpha(j)}|$, $x^{\alpha(3)} \neq 0$ and set $x = x^{\alpha(3)}$, $y = x^{\alpha(2)}$, $z = x^{\alpha(1)}$, $\varphi = \xi^2 = \arctan(\frac{y}{x})$ and

$$
\theta = \xi^1 = \begin{cases} \arccos(z) \text{ if } x \geq 0\\ 2\pi - \arccos(z) \text{ else.}\end{cases} \tag{4.8}
$$

Then we have

$$
\begin{aligned}
x =& \sin\theta\cos\varphi\\
y =& \sin\theta\sin\varphi\\
z =& \cos\theta
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
\frac{\partial x}{\partial \xi^1} &= \cos\theta\cos\varphi \\
\frac{\partial x}{\partial \xi^2} &= -\sin\theta\sin\varphi \\
\frac{\partial y}{\partial \xi^1} &= \cos\theta\sin\varphi \\
\frac{\partial y}{\partial \xi^2} &= \sin\theta\cos\varphi \\
\frac{\partial z}{\partial \xi^1} &= -\sin\theta \\
\frac{\partial z}{\partial \xi^2} &= 0.
\end{aligned} \tag{4.10}
$$

We have to stipulate what we understand for a given triangle T of $\Gamma_h(0)$ by a representation of a^{ij} and b^i in $\tilde{T}$ (where $\tilde{T}$ denotes the lift of T to $\Gamma(0)$) with respect to given orthonormal basis v_1, v_2 in T. For this purpose we let $\tilde{v}_1, \tilde{v}_2$ be the orthogonal projections of $\frac{\partial}{\partial \xi^1}x$ and $\frac{\partial}{\partial \xi^2}x$ onto the plane which contains T. Then we define β^l_k, $k, l = 1, 2$ by

$$
\begin{aligned}
\tilde{v}_1 &= \beta^1_1 v_1 + \beta^2_1 v_2 \\
\tilde{v}_2 &= \beta^1_2 v_1 + \beta^2_2 v_2
\end{aligned} \tag{4.11}
$$

then

$$
\tilde{b}^i = b^j \beta^i_j, \quad \tilde{a}^{ij} = a^{kl}\beta^i_k \beta^j_l \tag{4.12}
$$

are what we understand by evaluating a^{ij}, b^i with respect to v_1, v_2. Let P denote the plane containing T and let M be the matrix representation with respect to the standard basis in $\mathbb{R}^3$ of an (arbitrary) extension to $\mathbb{R}^3 \times \mathbb{R}^3$ of the bilinear form on $P \times P$ represented by the matrix $\tilde{a}^{ij}$ with respect to the basis v_1, v_2. Correspondingly we construct a vector $B \in \mathbb{R}^3$ from $\tilde{b}^i$. Then M and B are the coefficients which are compatible with the input format of DUNE.

In order to calculate the right-hand sides of the transformed equation for the exact solution $\tilde{u}$ (this means we evaluate the left-hand side of (3.13) for our special $\tilde{u}$) we use our coordinates (ξ^i) to calculate the summand

$$
g^{ij}(t)\nabla^{\Gamma(0)}_i \nabla^{\Gamma(0)}_j \tilde{u} \tag{4.13}
$$

which can be put together from

$$
\nabla^{\Gamma(0)}_i \nabla^{\Gamma(0)}_j \tilde{u} = \frac{\partial^2}{\partial \xi^i \partial \xi^j}\tilde{u} - \Gamma^k_{ij}(0)\frac{\partial}{\partial \xi^k}\tilde{u}. \tag{4.14}
$$

Note, that $\tilde{u} = e^{-6t}x_1x_2 \neq e^{-6t}xy$ in general (depending on α).

We use the mesh generator Gmsh, see [10], and for the implementation, more precisely, DUNE-FEM, a discretization module for solving PDEs which depends on DUNE.

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